

# Functions of $q$ -positive type

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## Abstract

In this paper we characterize the subspace of  $\mathcal{L}_{q,1,v}$  of function which are the  $q$ -Bessel Fourier transform of positive functions in  $\mathcal{L}_{q,1,v}$ . As application we give a  $q$ -version of the Bochner's theorem.

## 1 Introduction and Preliminaries

Given a positive finite Borel measure  $\mu$  on the real line  $\mathbb{R}$ , the Fourier transform  $Q$  of  $\mu$  is the continuous function

$$Q(x) = \int_{\mathbb{R}} e^{-itx} d\mu(t).$$

The function  $Q$  is a positive definite function, i.e for any finite list of complex numbers  $z_1, \dots, z_n$  and real numbers  $x_1, \dots, x_n$

$$\sum_{r=1}^n \sum_{l=1}^n z_r \bar{z}_l Q(x_r - x_l) \geq 0.$$

Bochner's theorem says the converse is true, i.e. every positive definite function  $Q$  is the Fourier transform of a positive finite Borel measure. In  $q$ -Fourier analysis, semelar phenomenon will appear. It is the subject of our article.

In the following we consider  $0 < q < 1$  and we adopt the standard conventional notations of [2]. We put

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

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the set of  $q$ -real numbers and for complex  $a$

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad n = 1 \dots \infty.$$

Jackson's  $q$ -integral (see [3]) in the interval  $[0, \infty[$  is defined by

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^\infty q^n f(q^n).$$

Let  $\mathcal{C}_{q,0}$  and  $\mathcal{C}_{q,b}$  denote the spaces of functions defined on  $\mathbb{R}_q^+$  continued at 0, which are respectively vanishing at infinity and bounded. These spaces are equipped with the topology of uniform convergence, and by  $\mathcal{L}_{q,p,v}$  the space of functions  $f$  defined on  $\mathbb{R}_q^+$  such that

$$\|f\|_{q,p,v} = \left[ \int_0^\infty |f(x)|^p x^{2v+1} d_q x \right]^{1/p} < \infty.$$

The  $q$ -exponential function is defined by

$$e(z, q) = \sum_{n=0}^\infty \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

The normalized Hahn-Exton  $q$ -Bessel function of order  $v > -1$  (see [5]) is defined by

$$j_v(z, q) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q, q)_n (q^{v+1}, q)_n} z^n.$$

The  $q$ -Bessel Fourier transform  $\mathcal{F}_{q,v}$  introduced in [1,4] as follow

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t,$$

where

$$c_{q,v} = \frac{1}{1 - q} \frac{(q^{2v+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

Define the  $q$ -Bessel translation operator as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t, \quad \forall x, y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,v,1}.$$

Recall that  $T_{q,x}^v$  is said positive if  $T_{q,x}^v f \geq 0$  for  $f \geq 0$ . In the following we tack  $q \in Q_v$  where

$$Q_v = \{q \in ]0, 1[, \quad T_{q,x}^v \text{ is positive for all } x \in \mathbb{R}_q^+\}.$$

The  $q$ -convolution product of both functions  $f, g \in \mathcal{L}_{q,1,v}$  is defined by

$$f *_q g(x) = c_{q,v} \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

In the end we denote by  $\mathcal{A}_{q,v}$  the  $q$ -Wiener algebra

$$\mathcal{A}_{q,v} = \{f \in \mathcal{L}_{q,1,v}, \quad \mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}\}.$$

The followings results in this sections was proved in [1].

**Proposition 1** *Let  $n, m \in \mathbb{Z}$  and  $n \neq m$ , then we have*

$$c_{q,v}^2 \int_0^\infty j_v(q^n x, q^2) j_v(q^m x, q^2) x^{2v+1} d_q x = \frac{q^{-2n(v+1)}}{1-q} \delta_{nm}.$$

**Proposition 2**

$$|j_v(q^n, q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^{2v+2}; q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0 \\ q^{n^2+(2v+1)n} & \text{if } n < 0 \end{cases}.$$

**Proposition 3** *The  $q$ -Bessel Fourier transform*

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \rightarrow \mathcal{C}_{q,0},$$

*satisfying*

$$\|\mathcal{F}_{q,v}(f)\|_{\mathcal{C}_{q,0}} \leq B_{q,v} \|f\|_{q,1,v},$$

*where*

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

**Theorem 1** *Given  $f \in \mathcal{L}_{q,1,v}$  then we have*

$$\mathcal{F}_{q,v}^2(f)(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

*If  $f \in \mathcal{L}_{q,1,v}$  and  $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}$  then*

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

**Proposition 4** *Let  $f \in \mathcal{L}_{q,1,v}$  then*

$$T_{q,x}^v f(y) = \int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z,$$

where

$$D_v(x, y, z) = c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(z, q^2) t^{2v+1} d_q t.$$

**Proposition 5** *Given two functions  $f, g \in \mathcal{L}_{q,v,1}$  then*

$$f *_q g \in \mathcal{L}_{q,v,1},$$

and

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

**Proposition 6** *The  $q$ -Gauss kernel*

$$G^v(x, t, q^2) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty} e\left(-\frac{q^{-2v}}{t}x^2, q^2\right),$$

satisfying

$$\mathcal{F}_{q,v} \{e(-ty^2, q^2)\} (x) = G^v(x, t, q^2),$$

and for all function  $f \in \mathcal{C}_{q,b}$

$$\lim_{a \rightarrow 0} c_{q,v} \int_0^\infty f(x) G^v(x, a^2, q^2) x^{2v+1} d_q x = f(0).$$

**Theorem 2** *Given  $1 < p, p', r \leq 2$  and*

$$\frac{1}{p} + \frac{1}{p'} - 1 = \frac{1}{r}.$$

*If  $f \in \mathcal{L}_{q,p,v}$  and  $g \in \mathcal{L}_{q,p',v}$  then*

$$f *_q g \in \mathcal{L}_{q,r,v}.$$

## 2 Functions of $q$ -positive type

**Definition 1** A function  $\phi$  is of  $q$ -positive type if

$$\phi \in \mathcal{C}_{q,b} \cap \mathcal{L}_{q,1,v}$$

and for any finite list of complex numbers  $z_1, \dots, z_n$  and  $q$ -real numbers  $x_1, \dots, x_n$

$$\sum_{r=1}^n \sum_{l=1}^n z_r \bar{z}_l T_{q,x_r}^v \phi(x_l) \geq 0. \quad (1)$$

**Proposition 7** Let  $\phi \in \mathcal{A}_{q,v}$  of  $q$ -positive type then  $\mathcal{F}_{q,v}\phi$  is of  $q$ -positive type.

**Proof.** From Proposition 3 and the definition of the  $q$ -Wiener algebra

$$\mathcal{F}_{q,v}(\phi) \in \mathcal{C}_{q,b} \cap \mathcal{L}_{q,1,v}.$$

On the other hand, with the inversion formula in Theorem 1 we get

$$T_{q,x}^v \mathcal{F}_{q,v}(\xi)(y) = \int_0^\infty j_v(tx, q^2) j_v(ty, q^2) t^{2v+1} \phi(t) d_q t,$$

then

$$\begin{aligned} \sum_{r=1}^n \sum_{l=1}^n z_r \bar{z}_l T_{q,x_r}^v \mathcal{F}_{q,v} \xi(x_l) &= c_{q,v} \int_0^\infty \left[ \sum_{r=1}^n \sum_{l=1}^n z_r \bar{z}_l j_v(x_r t, q^2) j_v(x_l t, q^2) \right] t^{2v+1} \phi(t) d_q t \\ &= c_{q,v} \int_0^\infty \left[ \sum_{r=1}^n z_r j_v(x_r t, q^2) \right] \overline{\left[ \sum_{l=1}^n z_l j_v(x_l t, q^2) \right]} t^{2v+1} \phi(t) d_q t \\ &= c_{q,v} \int_0^\infty \left| \sum_{r=1}^n z_r j_v(x_r t, q^2) \right|^2 t^{2v+1} \phi(t) d_q t \geq 0. \end{aligned}$$

This finish the proof. ■

**Proposition 8** If  $\phi$  is of  $q$ -positive type and  $f \in \mathcal{L}_{q,2,v}$  then

$$\phi *_q f \in \mathcal{L}_{q,2,v},$$

and

$$\langle \phi *_q f, f \rangle \geq 0.$$

**Proof.** From Theorem 2 we see that  $\phi *_q f \in \mathcal{L}_{q,2,v}$ . On the other hand

$$\begin{aligned}\langle \phi *_q f, f \rangle &= c_{q,v}^2 \int_0^\infty \left[ \int_0^\infty T_{q,x}^v \phi(y) f(y) y^{2v+1} d_q y \right] f(x) x^{2v+1} d_q x \\ &= (1-q)^2 c_{q,v}^2 \sum_{r=1}^\infty \sum_{l=1}^\infty q^{(2v+2)r} f(q^r) q^{(2v+2)l} f(q^l) T_{q,q^r}^v \phi(q^l) \geq 0.\end{aligned}$$

This finish the proof. ■

**Corollary 1** *If  $\phi$  is of  $q$ -positive type then*

$$\mathcal{F}_{q,v} \phi(x) \geq 0, \quad \forall x \in \mathbb{R}_q^+.$$

**Proof.** Given  $x \in \mathbb{R}_q^+$  and let

$$f_x : t \mapsto c_{q,v} j_v(xt, q^2),$$

then with Proposition 2 we see that  $f_x \in \mathcal{L}_{q,2,v}$  and by Proposition 1

$$\mathcal{F}_{q,v} f_x(y) = \delta_{q,v}(x, y),$$

which implies (see[1])

$$\langle \mathcal{F}_{q,v} \phi \times \mathcal{F}_{q,v} f_x, \mathcal{F}_{q,v} f_x \rangle = \mathcal{F}_{q,v} \phi(x) \delta_{q,v}(x, x) = \frac{1}{(1-q)x^{2v+2}} \mathcal{F}_{q,v} \phi(x).$$

From Proposition 8

$$\langle \mathcal{F}_{q,v} \phi \times \mathcal{F}_{q,v} f, \mathcal{F}_{q,v} f \rangle = \langle \phi *_q f, f \rangle \geq 0,$$

this leads to the result. ■

**Proposition 9** *If  $\phi$  is of  $q$ -positive type then  $\mathcal{F}_{q,v} \phi \in \mathcal{L}_{q,1,v}$ .*

**Proof.** From Proposition 6

$$\lim_{a \rightarrow 0} \int_0^\infty e(-a^2 x, q^2) \mathcal{F}_{q,v} \phi(x) x^{2v+1} d_q x = \lim_{a \rightarrow 0} c_{q,v} \int_0^\infty G^v(x, a^2, q^2) \phi(x) x^{2v+1} d_q x = \phi(0).$$

By the monotone convergence theorem and the preview corollary we see that

$$\int_0^\infty |\mathcal{F}_{q,v} \phi(x)| x^{2v+1} d_q x = \int_0^\infty \mathcal{F}_{q,v} \phi(x) x^{2v+1} d_q x = \phi(0).$$

This finish the proof. ■

**Corollary 2** *If  $\phi$  is of  $q$ -positive type then there exist a positive function  $\xi \in \mathcal{A}_{q,v}$  such that*

$$\phi(x) = \mathcal{F}_{q,v}\xi(x), \quad \forall x \in \mathbb{R}_q^+.$$

**Proof.** From the inversion formula in theorem 1

$$\phi(x) = \mathcal{F}_{q,v}^2\phi(x), \quad \forall x \in \mathbb{R}_q^+.$$

Define the function  $\xi$  as follows

$$\xi(x) = \mathcal{F}_{q,v}\phi(x).$$

By the use of Corollary 1 and Proposition 9 we see that  $\xi$  is a positive function of  $\mathcal{A}_{q,v}$ . ■

**Proposition 10** *Suppose  $\phi$  is of  $q$ -positive type. If  $f \in \mathcal{L}_{q,1,v}$  is positive function then the product  $\phi\mathcal{F}_{q,v}f$  is of  $q$ -positive type.*

**Proof.** Proposition 5 and Proposition 9 give

$$\mathcal{F}_{q,v}(\phi\mathcal{F}_{q,v}f)(t) = \mathcal{F}_{q,v}\phi *_q f(t), \quad \forall t \in \mathbb{R}_q^+,$$

then

$$\begin{aligned} & \sum_{r=1}^n \sum_{l=1}^n z_r \overline{z_l} T_{q,x_r}^v (\phi\mathcal{F}_{q,v}f)(x_l) \\ &= c_{q,v} \int_0^\infty \left[ \sum_{r=1}^n \sum_{l=1}^n z_r j_v(x_r t, q^2) \overline{z_l j_v(x_l t, q^2)} \right] \mathcal{F}_{q,v}(\phi\mathcal{F}_{q,v}f)(t) t^{2v+1} d_q t \\ &= c_{q,v} \int_0^\infty \left[ \sum_{r=1}^n \sum_{l=1}^n z_r j_v(x_r t, q^2) \overline{z_l j_v(x_l t, q^2)} \right] \mathcal{F}_{q,v}\phi *_q f(t) t^{2v+1} d_q t \\ &= c_{q,v} \int_0^\infty \left| \sum_{r=1}^n z_r j_v(x_r t, q^2) \right|^2 \mathcal{F}_{q,v}\phi *_q f(t) t^{2v+1} d_q t. \end{aligned}$$

From the definition of the  $q$ -convolution product we write

$$\mathcal{F}_{q,v}\phi *_q f(t) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}\phi(z) T_{q,t} f(z) z^{2v+1} d_q z.$$

Proposition 4 give

$$T_{q,t}f(z) = c_{q,v} \int_0^\infty D_v(t, z, s) f(s) s^{2v+1} d_q s \geq 0.$$

This implies with Corollary 1

$$\mathcal{F}_{q,v} \phi *_q f(t) \geq 0,$$

which leads to the result. ■

**Corollary 3** *Given two functions  $\phi_1, \phi_2$  which are of  $q$ -positive type then the product  $\phi_1 \times \phi_2$  is also of  $q$ -positive type.*

**Proof.** Let  $\xi = \mathcal{F}_{q,v} \phi_2$  then with the inversion formula in theorem 1 we see that  $\mathcal{F}_{q,v} \xi = \phi_2$ . Proposition 9 give

$$\xi \in \mathcal{L}_{q,1,v},$$

and by Proposition 10 we achieved the proof. ■

### 3 $q$ -Bochner's Theorem

We consider the set  $\mathcal{M}_q^+$  of positives and bonded measures on  $\mathbb{R}_q^+$ . The  $q$ -Bessel Fourier transform of  $\xi \in \mathcal{M}_q^+$  is defined by

$$\mathcal{F}_{q,v}(\xi)(x) = \int_0^\infty j_v(tx, q^2) t^{2v+1} d_q \xi(t).$$

The  $q$ -convolution product of two measures  $\xi, \rho \in \mathcal{M}_q^+$  is given by

$$\xi *_q \rho(f) = \int_0^\infty T_{q,x}^v f(t) t^{2v+1} d_q \xi(x) d_q \rho(t),$$

and we have

$$\mathcal{F}_{q,v}(\xi *_q \rho) = \mathcal{F}_{q,v}(\xi) \mathcal{F}_{q,v}(\rho).$$

The following Theorem (see[1]) is crucial for the proof of our main result.



**Theorem 3** *Let  $(\xi_n)_{n \geq 0}$  be a sequences of probability measures of  $\mathcal{M}_q^+$  such that*

$$\lim_{n \rightarrow \infty} \mathcal{F}_{q,v}(\xi_n)(x) = \psi(x),$$

*then there exists  $\xi \in \mathcal{M}_q^+$  such that the sequence  $\xi_n$  converge strongly toward  $\xi$  and*

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

In the following we consider the function  $\psi$  defined by

$$\psi(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Now we are in a position to state and prove the q-analogue of the Bochner's theorem

**Theorem 4** *Let  $\phi$  be a function defined on  $\mathbb{R}_q^+$  continued at 0. Assume that the following function*

$$\phi_n : x \mapsto \phi(x)\psi(q^n x),$$

*satisfy (1) for all  $n \in \mathbb{N}$  then there exist  $\xi \in \mathcal{M}_q^+$  such that*

$$\mathcal{F}_{q,v}(\xi) = \phi.$$

**Proof.** The function  $\phi_n$  is of  $q$ -positive type. From Corollary 2 there exist  $\varrho_n$  a positive function of  $\mathcal{A}_{q,v}$  such that

$$\mathcal{F}_{q,v}(\varrho_n) = \phi_n.$$

The measure  $\xi_n$  defined by

$$d_q \xi_n(x) = \varrho_n(x) d_q x,$$

belong to  $\mathcal{M}_q^+$  and

$$\int_0^\infty x^{2v+1} d_q \xi_n(x) = \mathcal{F}_{q,v}(\varrho_n)(0) = \phi_n(0) = \phi(0).$$

Assume that  $\phi(0) = 1$ . On the other hand

$$\lim_{n \rightarrow \infty} \mathcal{F}_{q,v}(\xi_n)(x) = \lim_{n \rightarrow \infty} \phi_n(x) = \phi(x).$$

From Theorem 3 there exists  $\xi \in \mathcal{M}_q^+$  such that the sequence  $\xi_n$  converge strongly toward  $\xi$ , and

$$\mathcal{F}_{q,v}(\xi) = \phi,$$

which leads to the result. ■

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